# The module of Kähler differentials

Eric Walker `cew028@uark.edu`

CARES

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# Outline

Topics:

- Why?
- Derivations
- Definition of the Kähler differentials
- Construction of the Kähler differentials
- The first fundamental exact sequence
- The second fundamental exact sequence
- Where do you go from here?

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- Why?
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- Construction of the Kähler differentials
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- The second fundamental exact sequence
- Where do you go from here?

Conventions:

- $k$ is a ring, and all rings are commutative and unital
- a $k$-algebra is a ring $A$ with a structure map $\varphi : k \to A$

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As algebraists, this is formal symbol moving, not $\varepsilon$-neighborhoods. (But will a geometric picture remain?)

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This is some of the cal 1 rules... sorta. Is it enough?

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In almost all contexts we will care about, $\varphi : k \to A$ is injective, so we will typically write $c$ for $\varphi(c)$, and then $\delta(cf) = c\delta(f)$.

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Not only that, but $\mathrm{Der}_k(A; M)$ is an $A$-submodule via the action $(f \cdot \delta)(g) = f\delta(g)$. We can add, subtract, and scale derivations.

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Let $\delta : A \to M$ be an $k$-linear derivation. There exists a unique $A$-module homomorphism $\Omega_{A/k} \to M$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[d, "\delta"'] & \Omega_{A/k} \arrow[dl, dashed, "\exists!"] \\ M & \end{tikzcd}$$

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That is, there is an isomorphism of $A$-modules

$$\mathrm{Hom}_A(\Omega_{A/k}, M) \cong \mathrm{Der}_k(A; M)$$

given by composition with the universal derivation $d : A \to \Omega_{A/k}$.

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But what about in general?

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- $d\varphi(c) = 0$.

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But this shouldn't necessarily sit well with us: where is the geometry?

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Let $d : A \to K'$ be defined by $d(f) = 1 \otimes f - f \otimes 1$.

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But why $I/I^2$? There's geometry here!
Thinking in terms of elements of a ring as functions, the module $I/I^2$ amounts to functions which vanish modulo vanishing to second order.
You can think: take a Taylor series and truncate it to get the first order differentiation. We'll see more geometry later!

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Let $M = Adx_1 \oplus \cdots \oplus Adx_n$. The partial derivative $\partial_i : A \to Adx_i$ is a derivation, so $\delta = \sum \partial_i$ is a derivation $A \to M$.

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Using the universal property, we get a unique $A$-module map $\psi$ such that the diagram commutes.

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[d, "\delta"'] & \Omega_{A/k} \arrow[dl, dashed, "\psi"] \\ A \cdot dx_1 \oplus \cdots \oplus A \cdot dx_n & \end{tikzcd}$$

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Using the universal property, we get a unique $A$-module map $\psi$ such that the diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "d"] \arrow[d, "\delta"'] & \Omega_{A/k} \arrow[dl, dashed, "\psi"] \\
A \cdot dx_1 \oplus \cdots \oplus A \cdot dx_n &
\end{tikzcd}$$

Let $\Omega_{A/k} \cong \bigoplus Adf / \sim$, which was our first construction.

# Construction of the Kähler differentials

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$\psi$ is injective: If $\psi(df) = 0$, then $\delta(f) = 0$, so $\partial_i(f) = 0 dx_i$ for all $i$. Thus $f$ is $x_i$-free, i.e., $f \in k$, so $df = 0$ in $\bigoplus Adf / \sim$.

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so $\{dx_1, \ldots, dx_n\} \subseteq \bigoplus Adf/\sim$ maps to the basis $\{1dx_1, \ldots, 1dx_n\}$ under $\psi$.

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**Theorem.** *Let $k \to R \to S$ be ring maps. The following sequence of $S$-modules is exact.*

$$\Omega_{R/k} \otimes_R S \to \Omega_{S/k} \to \Omega_{S/R} \to 0$$

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Proof.

[00RS]. □

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But $(f)$ isn't an $A$-module, it's a $k[x,y]$-module. So we base change to $A$.
Now the above sequence is exact.

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$$(f)/(f^2) \to \Omega_{k[x,y]/k} \otimes_{k[x,y]} A \to \Omega_{A/k} \to 0$$

What happens when $A \neq k[x,y]/(f)$? E.g., $k[x,y]/(f_1, \ldots, f_s)$? Or, more generally, ring maps $k \to R \twoheadrightarrow S$? The FFES gives us

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$$\left.{(f)}\middle/{(f^2)}\right. \to \Omega_{k[x,y]/k} \otimes_{k[x,y]} A \to \Omega_{A/k} \to 0$$

What happens when $A \neq k[x,y]/(f)$? E.g., $k[x,y]/(f_1, \ldots, f_s)$? Or, more generally, ring maps $k \to R \twoheadrightarrow S$? The FFES gives us

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**Theorem.** *Let $R \twoheadrightarrow S$ be a map of $k$-algs. Let $I = \ker(R \twoheadrightarrow S)$. The following sequence of $S$-modules is exact.*

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Proof.

[00RU]. $\square$

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**Corollary.** *If* $A \cong k[x_1, \ldots, x_n]/(f_1, \ldots, f_s)$, *then*

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Proof.

Let $R = k[x_1, \ldots, x_n]$, $S = A$, and observe that

$$df_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j.$$

$\square$

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3. If $A'' = k[x_1,\ldots,x_n]/(f_1,\ldots,f_s)$, then

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Suppose $(R, \mathfrak{m}, k)$ is a local ring, so it can be understood as in correspondence with a point $x$ of some LRS $X$. Using the map of $k$-algs $k \to R \twoheadrightarrow R/\mathfrak{m} = k$, we get

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But $\varphi$ is injective too! To see this, we'll use the fact that $\mathrm{Hom}(-, k)$ is left exact, and check that

$$\mathrm{Hom}_k(\Omega_{R/k} \otimes_R k, k) \xrightarrow{\varphi_*} \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$$

is surjective.

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Why $\mathrm{Hom}(-, k)$?!?! Because $\mathfrak{m}/\mathfrak{m}^2$ is the Zariski cotangent space at $x$, and its $k$-vector space dual $\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ is the tangent space, so it's reasonable to look at.

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Idea: To show $\varphi_*$ is surjective, we show any $k$-linear morphism $\psi : \mathfrak{m}/\mathfrak{m}^2 \to k$ lifts to $\Omega_{R/k} \otimes_R k \to k$. Define a map $R \to k$ by $r = a + b$ for $a \in k$ and $b \in \mathfrak{m}$; check that $r \mapsto \psi(b)$ is a derivation. Then show $\varphi_*$ is surjective via universal property of $\Omega_{R/k}$.

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So Kähler differentials tell us geometry! $\Omega_{R/k} \otimes_R k \cong \mathfrak{m}/\mathfrak{m}^2$ is the cotangent space.

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$\Omega_{A/k}$ is differentiation in module form. From last year's CARES: $J^1A$ is differentiation in $k$-algebra form. One might wonder: is there a connection? Yes! And it's exactly what you hope. The functor $\mathrm{Sym} : \mathbf{Mod}_k \to \mathbf{Alg}_k$ is the natural way to take a module to an algebra. And indeed,

$$J^1A \cong \mathrm{Sym}\,\Omega_{A/k}.$$

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The differential $d : \Omega^p_{A/k} \to \Omega^{p+1}_{A/k}$ satisfies $d^2 = 0$ and there is a multiplicative map $\Omega^p_{A/k} \otimes_A \Omega^q_{A/k} \to \Omega^{p+q}_{A/k}$, so we get a differential graded algebra / cochain complex $\Omega^\bullet_{A/k}$.

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$$h^i(\Omega^\bullet_{A/k}) := H^i_{dR}(A/k).$$

Connect this to Duncan's 15 Sept talk about the Koszul complex and Čech / sheaf cohomology!

# Where do you go from here?

Homological algebra and derived functors: we have two sequences which are exact on the right:

$$\#1: \; k \to R \to S \Rightarrow \Omega_{R/k} \otimes_R S \to \Omega_{S/k} \to \Omega_{S/R} \to 0.$$

$$\#2: \; k \to R \xrightarrow{\psi} S \Rightarrow {}^{\ker\psi}\!\big/_{\ker\psi^2} \to \Omega_{R/k} \otimes_R S \to \Omega_{S/k} \to 0.$$

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You might want to extend to long exact sequences. This is kinda funky since $\mathbf{Alg}_k$ is not an abelian category. But it can be done *homotopically*. You get something called the cotangent complex $\mathbf{L}_{A/k}$.

# Thank you!

Exact sequences. The Stacks project `https://stacks.math.columbia.edu` Tags: [00RS] [00RU]

Jet spaces. Jet schemes and singularities, Lawrence Ein & Mircea Mustaţă **Ex** 2.5

Homotopy and $\mathbf{L}_{A/k}$. An introduction to homological algebra, Charles Weibel §8.8.
DAG IV: Deformation Theory, Jacob Lurie