The module of Kähler differentials

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Outline

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Topics:

- Why?
- Derivations
- Definition of the Kähler differentials
- Construction of the Kähler differentials
- The first fundamental exact sequence
- The second fundamental exact sequence
- Where do you go from here?

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- Why?
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- The second fundamental exact sequence
- Where do you go from here?

Conventions:

- k is a ring, and all rings are commutative and unital
- a k-algebra is a ring A with a structure map $\varphi:k\to A$

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This is some of the cal 1 rules... sorta. Is it enough?

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- **2** Power Rule. Let n = 2.

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In almost all contexts we will care about, $\varphi : k \to A$ is injective, so we will typically write c for $\varphi(c)$, and then $\delta(cf) = c\delta(f)$.

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By the "**Constant Multiple Rule**," δ is a *k*-module homomorphism, so $\text{Der}_k(A; M) \subseteq \text{Hom}_k(A, M)$.

Not only that, but $\text{Der}_k(A; M)$ is an A-submodule via the action $(f \cdot \delta)(g) = f\delta(g)$. We can add, subtract, and scale derivations.

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Definition of the Kähler differentials **Definition.** The module of Kähler differentials of A over $k, \ \Omega_{A/k},$

 $\Omega_{A/k}$

$$A \xrightarrow{d} \Omega_{A/k}$$

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Let $\delta : A \to M$ be an k-linear derivation.

$$\begin{array}{c} A \xrightarrow{d} \Omega_{A/k} \\ \delta \\ M \end{array}$$

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Let $\delta: A \to M$ be an k-linear derivation. There exists a unique A-module homomorphism $\Omega_{A/k} \to M$ such that the following diagram commutes.

$$\begin{array}{ccc} A & \stackrel{d}{\longrightarrow} & \Omega_{A/k} \\ \downarrow & & \swarrow & \uparrow & \uparrow \\ M & & & \downarrow & & \uparrow & \uparrow \\ M & & & & & \downarrow & & \uparrow & \uparrow & \downarrow \\ \end{array}$$

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$$\begin{array}{ccc} A & \stackrel{d}{\longrightarrow} & \Omega_{A/k} \\ \downarrow & \downarrow^{& \swarrow} \\ M \end{array}$$

That is, there is an isomorphism of A-modules

$$\operatorname{Hom}_A(\Omega_{A/k}, M) \cong \operatorname{Der}_k(A; M)$$

given by composition with the universal derivation $d: A \to \Omega_{A/\underline{k}}$.

We should always have universal property concerns!

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But what about in general?

Here's the easiest way to build $\Omega_{A/k}$. Let

$$F = \bigoplus_{f \in A} A \cdot df.$$

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- $\bullet \ d(f+g)=df+dg,$
- $\bullet \ d(fg) = df \cdot g + f \cdot dg,$
- $d\varphi(c) = 0.$

Easy exercise: by construction, K along with the map $d: A \to K$ satisfies the universal property of $\Omega_{A/k}, d$.

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Idea: this is the formal symbol moving of calculus 1 students. Building K amounts to defining exactly the relations needed, and no more, that guarantee $d: A \to K$ is a derivation.

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But this shouldn't necessarily sit well with us: where is the geometry?

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Here's the second easiest way to build $\Omega_{A/k}$. Let

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Let $d: A \to K'$ be defined by $d(f) = 1 \otimes f - f \otimes 1$.

Medium exercise: by construction, K' along with the morphism $d: A \to K'$ satisfies the universal property of $\Omega_{A/k}, d$.

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But why I/I^2 ? There's geometry here!

Thinking in terms of elements of a ring as functions, the module I/I^2 amounts to functions which vanish modulo vanishing to second order.

You can think: take a Taylor series and truncate it to get the first order differentiation. We'll see more geometry later!

Let $A = k[x_1, \ldots, x_n]$. What is $\Omega_{A/k}$? We claim it is

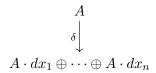
 $A \cdot dx_1 \oplus \cdots \oplus A \cdot dx_n.$

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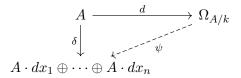
Let $M = Adx_1 \oplus \cdots \oplus Adx_n$. The partial derivative $\partial_i : A \to Adx_i$ is a derivation, so $\delta = \sum \partial_i$ is a derivation $A \to M$.



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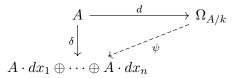
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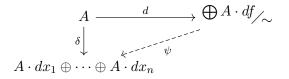
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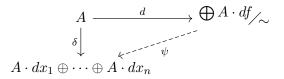
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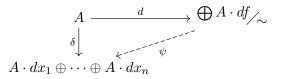
Let $\Omega_{A/k} \cong \bigoplus Adf / \sim$, which was our first construction.





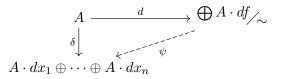
 ψ is injective: If $\psi(df) = 0$, then $\delta(f) = 0$, so $\partial_i(f) = 0dx_i$ for all *i*. Thus *f* is x_i -free, i.e., $f \in k$, so df = 0 in $\bigoplus Adf / \sim$.

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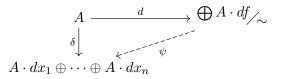
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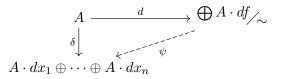
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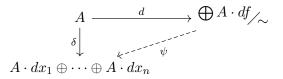
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$$\psi (dx_i) = \delta (x_i)$$
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so $\{dx_1, \ldots, dx_n\} \subseteq \bigoplus Adf / \sim$ maps to the basis $\{1dx_1, \ldots, 1dx_n\}$ under ψ .

$$f = \sum_{i,j} c_{ij} x^i y^j.$$

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$$df = \sum_{i,j} c_{ij} d\left(x^{i} y^{j}\right) = \sum_{i,j} c_{ij} \left(d\left(x^{i}\right) y^{j} + x^{i} d\left(y^{j}\right)\right)$$

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$$= \sum_{i,j} c_{ij} \left(i x^{i-1} y^{j} dx + x^{i} j y^{j-1} dy\right)$$

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Another example (goal: to generalize)! Let $A = k[x, y]/(y - x^2)$. If $f \in A$, then we can represent

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$$= \sum_{i,j} c_{ij} \left(i x^{i-1} y^{j} dx + x^{i} j y^{j-1} dy\right)$$

So $\Omega_{A/k}$ is generated as an A-module by dx and dy.

Another example (goal: to generalize)! Let $A = k[x, y]/(y - x^2)$. If $f \in A$, then we can represent

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is an isomorphism. But that doesn't generalize.

Theorem. Let $k \to R \to S$ be ring maps. The following sequence of S-modules is exact.

$$\Omega_{R/k} \otimes_R S \to \Omega_{S/k} \to \Omega_{S/R} \to 0$$

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Proof.

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Now the above sequence is exact.

$$(f)_{(f^2)} \to \Omega_{k[x,y]/k} \otimes_{k[x,y]} A \to \Omega_{A/k} \to 0$$

What happens when $A \neq k[x, y]/(f)$? E.g., $k[x, y]/(f_1, \dots, f_s)$? Or, more generally, ring maps $k \to R \twoheadrightarrow S$? The FFES gives us

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Repeat the same argument as before. We get: **Theorem.** Let $R \rightarrow S$ be a map of k-algs. Let $I = \ker(R \rightarrow S)$. The following sequence of S-modules is exact.

$$I \not/_{I^2} \xrightarrow{f \mapsto df \otimes 1} \Omega_{R/k} \otimes_R S \to \Omega_{S/k} \to 0$$

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Proof. [00RU].

$$I_{I^2} \xrightarrow{f \mapsto df \otimes 1} \Omega_{R/k} \otimes_R S \to \Omega_{S/k} \to 0$$

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Corollary. If $A \cong k[x_1, \ldots, x_n]/(f_1, \ldots, f_s)$, then

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Corollary. If $A \cong k[x_1, \dots, x_n]/(f_1, \dots, f_s)$, then $\Omega_{A/k} \cong \operatorname{coker} \left[\frac{\partial f_i}{\partial x_i}\right].$

Proof.

Let $R = k[x_1, \ldots, x_n]$, S = A, and observe that

$$df_i = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} dx_j.$$

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2 If
$$A' = k[x, y, z]/(xy, xz, yz)$$
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3 If
$$A'' = k[x_1, \dots, x_n]/(f_1, \dots, f_s)$$
, then

$$\Omega_{A''/k} \cong \frac{A''dx_1 \oplus \cdots A''dx_n}{(df_1, \dots, df_s)}.$$

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Suppose (R, \mathfrak{m}, k) is a local ring, so it can be understood as in correspondence with a point x of some LRS X. Using the map of k-algs $k \to R \twoheadrightarrow R/\mathfrak{m} = k$, we get

$$\mathfrak{m}_{\mathfrak{m}^2} \xrightarrow{\varphi} \Omega_{R/k} \otimes_R k \to \Omega_{k/k} = 0$$

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But φ is injective too! To see this, we'll use the fact that Hom(-, k) is left exact, and check that

$$\operatorname{Hom}_k(\Omega_{R/k} \otimes_R k, k) \xrightarrow{\varphi_*} \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$$

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is surjective.

$$\mathfrak{m}_{\mathfrak{m}^2} \xrightarrow{\varphi} \Omega_{R/k} \otimes_R k \to 0$$

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Why $\operatorname{Hom}(-, k)$?!?!



$$\mathfrak{m}_{\mathfrak{m}^{2}} \xrightarrow{\varphi} \Omega_{R/k} \otimes_{R} k \to 0$$
$$0 \to \operatorname{Hom}_{k}(\Omega_{R/k} \otimes_{R} k, k) \xrightarrow{\varphi_{*}} \operatorname{Hom}_{k}(\mathfrak{m}/\mathfrak{m}^{2}, k)$$

Why Hom(-, k)?!?! Because $\mathfrak{m}/\mathfrak{m}^2$ is the Zariski cotangent space at x, and its k-vector space dual Hom $_k(\mathfrak{m}/\mathfrak{m}^2, k)$ is the tangent space, so it's reasonable to look at.

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Why Hom(-, k)?!?! Because $\mathfrak{m}/\mathfrak{m}^2$ is the Zariski cotangent space at x, and its k-vector space dual Hom $_k(\mathfrak{m}/\mathfrak{m}^2, k)$ is the tangent space, so it's reasonable to look at.

Idea: To show φ_* is surjective, we show any k-linear morphism $\psi : \mathfrak{m}/\mathfrak{m}^2 \to k$ lifts to $\Omega_{R/k} \otimes_R k \to k$. Define a map $R \to k$ by r = a+b for $a \in k$ and $b \in \mathfrak{m}$; check that $r \mapsto \psi(b)$ is a derivation. Then show φ_* is surjective via universal property of $\Omega_{R/k}$.

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 $\Omega_{A/k}$ is differentiation in module form. From last year's CARES: J^1A is differentiation in k-algebra form. One might wonder: is there a connection? Yes! And it's exactly what you hope. The functor Sym : $\mathbf{Mod}_k \to \mathbf{Alg}_k$ is the natural way to take a module to an algebra. And indeed,

 $J^1 A \cong \operatorname{Sym} \Omega_{A/k}.$

You can also build out differential forms à la Calculus 3 in the natural way. Let $\Omega^p_{A/k}$ be the *p*th exterior power of $\Omega_{A/k}$ in the category of A-modules.

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The differential $d: \Omega^p_{A/k} \to \Omega^{p+1}_{A/k}$ satisfies $d^2 = 0$ and there is a multiplicative map $\Omega^p_{A/k} \otimes_A \Omega^q_{A/k} \to \Omega^{p+q}_{A/k}$, so we get a differential graded algebra / cochain complex $\Omega^{\bullet}_{A/k}$.

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Connect this to Duncan's 15 Sept talk about the Koszul complex and Čech / sheaf cohomology!

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Homological algebra and derived functors: we have two sequences which are exact on the right:

$$\begin{split} &\#1: k \to R \to S \Rightarrow \Omega_{R/k} \otimes_R S \to \Omega_{S/k} \to \Omega_{S/R} \to 0. \\ &\#2: k \to R \xrightarrow{\psi} S \Rightarrow \overset{\text{ker}}{\longrightarrow} \psi'_{\text{ker}} \psi^2 \to \Omega_{R/k} \otimes_R S \to \Omega_{S/k} \to 0. \end{split}$$

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You might want to extend to long exact sequences. This is kinda funky since \mathbf{Alg}_k is not an abelian category. But it can be done *homotopically*. You get something called the cotangent complex $\mathbf{L}_{A/k}$.

Thank you!

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Exact sequences. The Stacks project https://stacks.math.columbia.edu Tags: [00RS] [00RU]

Jet spaces. Jet schemes and singularities, Lawrence Ein & Mircea Mustață $\mathbf{Ex}~2.5$

Homotopy and $\mathbf{L}_{A/k}$. An introduction to homological algebra, Charles Weibel §8.8. DAG IV: Deformation Theory, Jacob Lurie